Discontinuous distributions in thermal plasmas

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Summary. — We develop a new method for describing the dynamics of 3-dimensional thermal plasmas. Using a piecewise constant 1-particle distribution, we reduce the Vlasov equation to a generalized Lorentz force equation for a family of vector fields encoding the discontinuity. By applying this equation to longitudinal electrostatic plasma oscillations, and coupling it to Maxwell's equations, we obtain a limit on the magnitude of the electric field in relativistic thermal plasma oscillations. We derive an upper bound on the limit and discuss its applicability in a background magnetic field.

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1. – Introduction

High-power lasers and plasmas may be used to accelerate electrons by electric fields that are orders of magnitude greater than those achievable using conventional methods [1]. An intense laser pulse is used to drive a wave in a plasma and, for sufficiently large fields, non-linearities lead to collapse of the wave structure ('wave-breaking') due to sufficiently large numbers of electrons becoming trapped in the wave.

Hydrodynamic investigations of wave-breaking were first undertaken for cold plasmas [2, 3] and thermal effects were later included in non-relativistic [4] and relativistic contexts [5-7] (see [8] for a discussion of the numerous approaches). However, it is clear that the value of the electric field at which the wave breaks (the electric field's 'wave-breaking limit') is highly sensitive to the details of the hydrodynamic model.

Plasmas dominated by collisions are described by a pressure tensor that does not deviate far from isotropy, whereas an intense and ultrashort laser pulse propagating through an underdense plasma will drive the plasma anisotropically over typical acceleration timescales. Thus, it is important to accommodate 3-dimensionality and allow for anisotropy when investigating wave-breaking limits.

Our aim is to uncover the relationship between wave-breaking limits and the shape of the 1-particle distribution f. In general, the detailed structure of f cannot be reconstructed from a few low-order moments so we adopt a different approach based on a

particular class of piecewise constant 1-particle distributions. Our approach may be considered as a multi-dimensional generalization of the 1-dimensional relativistic 'waterbag' model employed in [5] (for a discussion of the relationship between our approach and [5] see [9]).

We employ the Einstein summation convention throughout and units are used in which the speed of light $c = 1$ and the permittivity of the vacuum $\varepsilon_0 = 1$. Lowercase Latin indices a, b, c run over $0, 1, 2, 3$.

2. – Vlasov-Maxwell system

Our attention is focussed on plasmas evolving over timescales during which the 'discrete' nature (collisions) of the plasma electrons can be neglected and the plasma ions can be prescribed as a background. Such configurations are well described by the covariant Vlasov-Maxwell system [10, 11] which, for the purposes of this paper, is most usefully expressed in the language of exterior calculus (see, for example, [12, 13]). We will now briefly summarize the particular formulation of the Vlasov-Maxwell system employed here.

Let $(\mathcal{M}, g)$ be a spacetime with signature $(-, +, +, +)$ for the metric tensor g. Each point $p \in \mathcal{M}$ is associated with a space $\mathcal{E}_p \subset T_p\mathcal{M}$ of future-directed unit normalized vectors on $\mathcal{M}$,

$$\mathcal{E}_p = \{(x(p), \dot{x}) \in T_p\mathcal{M} : g_{ab}(x(p))\dot{x}^a\dot{x}^b = -1 \text{ and } \dot{x}^0 > 0\}, \tag{1}$$

where g_{ab} are the components of the metric g in a coordinate system (x^a) whose patch contains p and $(x^a, \dot{x}^b)$ are induced coordinates on $T\mathcal{M}$. The total space $\mathcal{E}$ of the bundle $(\mathcal{E}, \Pi, \mathcal{M})$ is the union of $\mathcal{E}_p$ over $p \in \mathcal{M}$ and Π is the restriction to $\mathcal{E}$ of the canonical projection on $T\mathcal{M}$.

Naturally induced tensors on $T\mathcal{M}$ include the dilation vector field X

$$X = \dot{x}^a \partial_a^{\boldsymbol{V}}, \tag{2}$$

the vertical lift $\star 1^{\boldsymbol{V}}$ of the volume 4-form $\star 1$ from $\mathcal{M}$ to $T\mathcal{M}$ and the horizontal 4-form $\#1$

$$\#1 = \sqrt{|\det \mathfrak{g}|}^{\boldsymbol{V}} dx^{0\boldsymbol{H}} \wedge dx^{1\boldsymbol{H}} \wedge dx^{2\boldsymbol{H}} \wedge dx^{3\boldsymbol{H}} \tag{3}$$

where $dx^{a\boldsymbol{H}}$ is the horizontal lift of dx^a from $\mathcal{M}$ to $T\mathcal{M}$ (see Appendix A for further details) and $\mathfrak{g} = (g_{ab})$ is the matrix of components of g.

The Vlasov-Maxwell system for f (a scalar field on $T\mathcal{M}$ whose restriction to $\mathcal{E}$ is the plasma electron 1-particle distribution) and the electromagnetic field F may be written

$$Lf \simeq 0, \tag{4}$$

$$dF = 0, \qquad d \star F = q \star (\widetilde{N}_{\text{ion}} - \widetilde{N}) \tag{5}$$

where $\simeq$ indicates equality on restriction by pullback from $T\mathcal{M}$ to $\mathcal{E}$ and

$$L = \dot{x}^a(\partial_a^{\boldsymbol{H}} + \mathbf{f}_a^{\boldsymbol{V}}) \in TT\mathcal{M}, \tag{6}$$

$$\mathbf{f}_a = -\frac{q}{m} F^b{}_a \partial_b \in T\mathcal{M} \tag{7}$$

with m the mass and q the charge of the electron ($q < 0$) and $F^a{}_b = g^{ac}F_{cb}$ the components of the electromagnetic 2-form $F = \frac{1}{2}F_{ab}dx^a \wedge dx^b$ on $\mathcal{M}$. The metric dual of a vector V is defined by $\widetilde{V}(Y) = g(Y, V)$ for all vectors Y on $\mathcal{M}$ and $\star$ is the Hodge map induced from the volume 4-form $\star 1$

$$\star 1 = \sqrt{|\det \mathfrak{g}|}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{8}$$

on $\mathcal{M}$. The components of the electron number 4-current $N = N^a(x(p))\partial_a$ at $p \in \mathcal{M}$ are given as an integral over the fibre $\mathcal{E}_p = \Pi^{-1}(p)$

$$\begin{aligned} N^a(x(p)) &= \int_{\Pi^{-1}(p)} \dot{x}^a f \iota_X \# 1 \\ &= -\int_{\Pi^{-1}(p)} \dot{x}^a f(x(p), \dot{x}) \frac{\sqrt{|\det \mathfrak{g}(x(p))|}}{g_{0c}(x(p))\dot{x}^c} d\dot{x}^1 \wedge d\dot{x}^2 \wedge d\dot{x}^3, \end{aligned} \tag{9}$$

and the ion number 4-current N_{ion} is prescribed as data.

The measure on $\mathcal{E}_p$ in (9) is induced from the 3-form $\iota_X \# 1$,

$$\begin{aligned} \iota_X \# 1 &= \sqrt{|\det \mathfrak{g}|}^{\boldsymbol{V}} \frac{1}{3!} \dot{x}^a \epsilon_{abcd} dx^{b\boldsymbol{H}} \wedge dx^{c\boldsymbol{H}} \wedge dx^{d\boldsymbol{H}} \\ &\simeq -\frac{\sqrt{|\det \mathfrak{g}|}^{\boldsymbol{V}}}{g_{0c}^{\boldsymbol{V}} \dot{x}^c} dx^{1\boldsymbol{H}} \wedge dx^{2\boldsymbol{H}} \wedge dx^{3\boldsymbol{H}} \end{aligned} \tag{10}$$

where ϵ_{abcd} is the alternating symbol with $\epsilon_{0123} = 1$.

The Vlasov-Maxwell equations constitute a non-linear integro-differential system. Direct calculation of its solutions for general plasma configurations is difficult and, to proceed analytically, it is common to approximate the above as a finite number of moments of f in $\dot{x}^a$ satisfying a non-linear field system on $\mathcal{M}$ (a so-called 'fluid' model). However, there are difficult issues associated with closing the resulting field system (see, for example, [14]) so we opt for a different approach. Our strategy is to reduce the system by employing a discontinuous f, and to proceed we need to cast (4) as an integral.

One may rewrite (4) as

$$d(f\omega) \simeq 0, \tag{11}$$

$$\omega = \iota_L(\star 1^{\boldsymbol{V}} \wedge \iota_X \# 1) \in \Lambda_6 T\mathcal{M}. \tag{12}$$

Integrating (11) over a 7-chain $\mathcal{A} \subset \mathcal{E}$ and applying Stokes's theorem yields

$$\int_{\partial\mathcal{A}} f\omega = 0, \tag{13}$$

with $\partial\mathcal{A}$ the boundary of $\mathcal{A}$. For differentiable distributions, this equation is equivalent to (11); however, since it makes no reference to the differentiability of f, it may be regarded as a generalisation of (11) applicable to discontinuous f.

3. – Evolution of discontinuities

Equation (13) may be used to develop an equation of motion for a discontinuity, which we choose as a local hypersurface $\mathcal{H}$. Suppose that $\mathcal{A}$ in (13) is a 7-dimensional 'pill-box' straddling $\mathcal{H}$. We may write $\partial\mathcal{A} = \sigma_+ + \sigma_- + \sigma_0$ where σ_+ and σ_- are the 'top' and 'bottom' of the pill-box and σ_0 is the 'sides' of the pill-box. Thus, in the limit as the volume of $\mathcal{A}$ tends to zero with σ_+ tending to σ and σ_- tending to $-\sigma$, we recover the condition

$$[f]\sigma^*\omega = 0, \tag{14}$$

where the image of σ is in $\mathcal{H}$ and $[f] = \sigma_+^* f + \sigma_-^* f$. Thus it follows that a finite discontinuity in f can occur only across the image of a chain Σ satisfying

$$\Sigma^*\omega = 0. \tag{15}$$

Suppose that Σ may be written locally

$$\begin{aligned} \Sigma :\ & \mathcal{V} \times \mathcal{D} \to \mathcal{E} \subset T\mathcal{M} \\ & (x^a, \xi^1, \xi^2) \mapsto (x^a, \dot{x}^b = \dot{\Sigma}^b(x, \xi)) \end{aligned} \tag{16}$$

for $\mathcal{V} \subset \mathcal{M}$, where $\dot{\Sigma}^b$ denotes the $\dot{x}^b$ component of Σ, and $(\xi^1, \xi^2) \in \mathcal{D} \subset \mathbb{R}^2$. It is then possible to translate (15) into a field equation for a family of vector fields V_ξ on $\mathcal{V}$ given as

$$V_\xi(p) = V_\xi^a(x(p))\partial_a = \dot{\Sigma}^a(x(p), \xi)\partial_a \tag{17}$$

where, since $g_{ab}(x(p))\dot{x}^a\dot{x}^b = -1$ at $p \in \mathcal{E}$, it follows

$$g(V_\xi, V_\xi) = -1. \tag{18}$$

Using (12, 15) it follows

$$\Sigma^*(\underbrace{\iota_L \star 1^{\boldsymbol{V}} \wedge \iota_X \#1}_{(a)} + \underbrace{\star 1^{\boldsymbol{V}} \wedge \iota_L \iota_X \#1}_{(b)}) = 0. \tag{19}$$

Consider first the term (a) in equation (19):

$$\Sigma^*(\iota_L \star 1^{\boldsymbol{V}} \wedge \iota_X \#1) = \Sigma^*(\dot{x}^a \iota_{\underset{a}{\partial^{\boldsymbol{H}}}} \star 1^{\boldsymbol{V}} \wedge \iota_X \#1) \tag{20}$$

where (6) and $\iota_{\underset{a}{\mathbf{f}^{\boldsymbol{V}}}} \star 1^{\boldsymbol{V}} = 0$ have been used (see (A.3) in Appendix A). Thus

$$\Sigma^*(\dot{x}^a \iota_{\underset{a}{\partial^{\boldsymbol{H}}}} \star 1^{\boldsymbol{V}} \wedge \iota_X \#1) = \star\widetilde{V}_\xi \wedge \Sigma^* \iota_X \#1, \tag{21}$$

since $\iota_{\underset{a}{\partial^{\boldsymbol{H}}}} \star 1^{\boldsymbol{V}} = (g_{ab} \star dx^b)^{\boldsymbol{V}}$ (see (A.10) in Appendix A). Furthermore, using (A.5) in Appendix A, it follows

$$\Sigma^*(dx^{a\boldsymbol{H}}) = DV_\xi^a + \underline{d}\dot{\Sigma}^a \tag{22}$$

where D is the exterior covariant derivative on $\mathcal{M}$ and $\underline{d}$ is the exterior derivative on $\mathcal{D}$. Using (10, 22) it follows

$$\Sigma^*\iota_X\#1 = \sqrt{|\det\mathfrak{g}|}\frac{1}{3!}\epsilon_{abcd}V_\xi^a(DV_\xi^b + \underline{d}\dot{\Sigma}^b)\wedge(DV_\xi^c + \underline{d}\dot{\Sigma}^c)\wedge(DV_\xi^d + \underline{d}\dot{\Sigma}^d) \tag{23}$$

and (20) is

$$\Sigma^*(\iota_L \star 1^{\boldsymbol{V}} \wedge \iota_X\#1) = \star\widetilde{V}_\xi \wedge \frac{1}{2!}\sqrt{|\det\mathfrak{g}|}\epsilon_{abcd}V_\xi^a DV_\xi^b \wedge \underline{d}\dot{\Sigma}^c \wedge \underline{d}\dot{\Sigma}^d \tag{24}$$

since $\star\widetilde{V}_\xi \wedge DV_\xi^a \wedge DV_\xi^b = 0$ ($\dim(\mathcal{M}) = 4$) and $\underline{d}\dot{\Sigma}^a \wedge \underline{d}\dot{\Sigma}^b \wedge \underline{d}\dot{\Sigma}^c = 0$ ($\dim(\mathcal{D})$=2).

Since $\star\widetilde{V}_\xi \wedge DV_\xi^b = -(\nabla_{V_\xi}V_\xi)^b \star 1$ and $\sqrt{|\det\mathfrak{g}|}\epsilon_{abcd} = \iota_{\partial_d}\iota_{\partial_c}\iota_{\partial_b}\iota_{\partial_a}\star 1$, it follows (24) may be written

$$\Sigma^*(\iota_L \star 1^{\boldsymbol{V}} \wedge \iota_X\#1) = \widetilde{V}_\xi \wedge \nabla_{V_\xi}\widetilde{V}_\xi \wedge \Omega_\xi \wedge d\xi^1 \wedge d\xi^2, \tag{25}$$

where the family of 2-forms Ω_ξ on $\mathcal{V}$ is

$$\Omega_\xi = \frac{\partial\dot{\Sigma}^a}{\partial\xi^1}\frac{\partial\dot{\Sigma}^b}{\partial\xi^2} g_{ac}\, g_{bd}\, dx^c \wedge dx^d. \tag{26}$$

The second term (b) in (19) can be rewritten using a similar procedure :

$$\Sigma^*(\star 1^{\boldsymbol{V}} \wedge \iota_L\iota_X\#1) = \star 1 \wedge \Sigma^*(\iota_L\iota_X\#1) \tag{27}$$

and from (6, 7, 10) it follows

$$\iota_L\iota_X\#1 = -\frac{1}{2!}\frac{q}{m}\sqrt{|\det\mathfrak{g}|}^{\boldsymbol{V}} F^{b\boldsymbol{V}}{}_e\,\dot{x}^a\dot{x}^e\epsilon_{abcd}dx^{c\boldsymbol{H}} \wedge dx^{d\boldsymbol{H}}. \tag{28}$$

where (A.9) and (A.10) have been used. Then

$$\Sigma^*(\iota_L\iota_X\#1) = -\frac{1}{2!}\frac{q}{m}\sqrt{|\det\mathfrak{g}|}F^b{}_e V_\xi^a V_\xi^e\epsilon_{abcd}(DV_\xi^c + \underline{d}\dot{\Sigma}^c)\wedge(DV_\xi^d + \underline{d}\dot{\Sigma}^d) \tag{29}$$

and

$$\Sigma^*(\star 1^{\boldsymbol{V}} \wedge \iota_L\iota_X\#1) = -\frac{q}{m}\widetilde{V}_\xi \wedge \iota_{V_\xi}F \wedge \Omega_\xi \wedge d\xi^1 \wedge d\xi^2. \tag{30}$$

Combining (25, 30) and (19) yields

$$\Sigma^*\omega = \widetilde{V}_\xi \wedge (\nabla_{V_\xi}\widetilde{V}_\xi - \frac{q}{m}\iota_{V_\xi}F) \wedge \Omega_\xi \wedge d\xi^1 \wedge d\xi^2 = 0. \tag{31}$$

Acting on (31) successively with ι_{V_ξ}, $\iota_{\partial/\partial\xi^1}$ and $\iota_{\partial/\partial\xi^2}$, and noting that $\iota_{V_\xi}\Omega_\xi = 0$ and $g(V_\xi, V_\xi) = -1$, yields

$$(\nabla_{V_\xi}\widetilde{V}_\xi - \frac{q}{m}\iota_{V_\xi}F) \wedge \Omega_\xi = 0. \tag{32}$$

Thus, solutions to (32) may be obtained by demanding that V_ξ is driven by the Lorentz force

$$\nabla_{V_\xi}\widetilde{V}_\xi = \frac{q}{m}\iota_{V_\xi}F. \tag{33}$$

However, although (33) is simpler than (32), there are simple solutions to (32) that do not satisfy (33); we will return to this point shortly.

4. – Non-linear electrostatic oscillations

A laser pulse travelling through a plasma can excite plasma oscillations, which induce very high longitudinal electric fields. Due to nonlinear effects, there is a maximum amplitude of electric field (the 'wave-breaking limit') that can be sustained in the plasma and, as mentioned in the introduction, our aim is to investigate the relationship between the shape of the distribution (i.e. Σ) and the wave-breaking limit.

For simplicity, we choose to describe the plasma using a distribution f where $f = \alpha$ is a positive constant inside a 7-dimensional region $\mathcal{U} \subset \mathcal{E}$ and $f = 0$ outside. In particular, we consider $\mathcal{U}$ to be the union over each point $p \in \mathcal{M}$ of a domain $\mathcal{W}_p$ whose boundary $\partial\mathcal{W}_p$ in $\mathcal{E}_p$ is topologically equivalent to the 2-sphere. Such distributions are sometimes called 'waterbags' in the literature and can be completely characterized by V_ξ and the constant α. Our approach may be considered as a multi-dimensional generalization of the purely 1-dimensional relativistic waterbag model used in [5] to examine wave-breaking.

We work in Minkowski spacetime $(\mathcal{M}, g)$ and assume that the ions constitute a homogeneous immobile background. We employ an inertial coordinate system (x^a) adapted to the ions:

$$N_{\text{ion}} = n_{\text{ion}}\partial_0, \tag{34}$$

where

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \tag{35}$$

and the ion proper number density n_{ion} is constant.

To proceed further we seek a form for Σ axisymmetric about $\dot{x}^3$ whose pointwise dependence in $\mathcal{M}$ is on the wave's phase $\zeta = x^3 - vx^0$ only, where v is constant and $0 < v < 1$. We suppose that all electrons described by f are travelling slower than the wave, and the wave 'breaks' if the longitudinal velocity of any plasma electron equals v (i.e. an electron 'catches up' with the wave).

Introduce

$$\boldsymbol{e}^1 = vdx^3 - dx^0, \qquad \boldsymbol{e}^2 = dx^3 - vdx^0, \tag{36}$$

and decompose $\widetilde{V}_\xi$ as

$$\widetilde{V}_\xi = [\mu(\zeta) + A(\xi^1)]\,\boldsymbol{e}^1 + \psi(\xi^1,\zeta)\,\boldsymbol{e}^2 + R\sin(\xi^1)\cos(\xi^2)dx^1 + R\sin(\xi^1)\sin(\xi^2)dx^2 \tag{37}$$

for $0 < \xi^1 < \pi$, $0 \le \xi^2 < 2\pi$ where $R > 0$ is constant.

Here, $(\gamma \boldsymbol{e}^1, \gamma \boldsymbol{e}^2, dx^1, dx^2)$, with $\gamma = 1/\sqrt{1-v^2}$, is an orthonormal coframe on $\mathcal{M}$ adapted to ζ. Since V_ξ is future-directed and timelike, and $\boldsymbol{e}^1$ is timelike, it follows $\boldsymbol{e}^1(V_\xi) < 0$ and $\mu + A(\xi^1) > 0$.

The component ψ is determined using (18),

$$\psi = -\sqrt{[\mu + A(\xi^1)]^2 - \gamma^2[1 + R^2 \sin^2(\xi^1)]}, \tag{38}$$

where the negative square root is chosen because no electron is moving faster along x^3 than the wave.

Substituting the *ansatz* (37) together with a purely longitudinal electric field depending only on ζ,

$$F = E(\zeta)\, dx^0 \wedge dx^3, \tag{39}$$

into (33) yields

$$E = \frac{1}{\gamma^2} \frac{m}{q} \frac{d\mu}{d\zeta}. \tag{40}$$

Equation (40) is used to eliminate E from Maxwell equations (5) and obtain a differential equation for μ.

The electron number current is calculated using (9):

$$N(p) = \alpha \bigg(\int_{\mathcal{W}_p} \dot{x}^a \iota_X \#1 \bigg) \partial_a \tag{41}$$

and (5, 37, 38) yield

$$\begin{aligned} \frac{1}{\gamma^2}\frac{d^2\mu}{d\zeta^2} =& -\frac{q^2}{m} n_{\mathrm{ion}} \gamma^2 \\ & -\frac{q^2}{m} 2\pi R^2 \alpha \int_0^\pi \Big([\mu + A(\xi^1)]^2 \; - \gamma^2[1 + R^2 \sin^2(\xi^1)] \Big)^{1/2} \sin(\xi^1)\, \cos(\xi^1)\, d\xi^1 \end{aligned} \tag{42}$$

and

$$2\pi R^2 \int_0^\pi A(\xi^1)\, \sin(\xi^1)\, \cos(\xi^1)\, d\xi^1 = -\frac{n_{\mathrm{ion}}\gamma^2\, v}{\alpha} \tag{43}$$

where α is the value of f inside $\mathcal{W}_p$.

The form of the 2nd order autonomous non-linear differential equation (42) for μ is fixed by specifying the generator $A(\xi^1)$ of $\partial\mathcal{W}_p$ subject to the normalization condition (43).

4·1. *Electrostatic wave-breaking.* – The form of the integrand in (42) ensures that the magnitude of oscillatory solutions to (42) cannot be arbitrarily large. For our model, the wave-breaking value μ_{wb} is the largest μ for which the argument of the square root in (42) vanishes,

$$\mu_{\mathrm{wb}} = \max\left\{ -A(\xi^1) + \gamma\sqrt{1 + R^2\sin^2(\xi^1)} \,\middle|\, 0 \le \xi^1 \le \pi \right\}, \tag{44}$$

because $\mu < \mu_{\mathrm{wb}}$ yields an imaginary integrand in (42) for some ξ^1. The positive square root in (44) is chosen because, as discussed above, $\mu + A(\xi^1) > 0$ and in particular $\mu_{\mathrm{wb}} + A(\xi^1) > 0$.

The wave-breaking limit $E_{\max}$ is obtained by evaluating the first integral of (42) between μ_{wb} where E vanishes and the equilibrium[1] value μ_{eq} of μ where E is at a maximum. Using (43) to eliminate α it follows that μ_{eq} satisfies

$$\begin{aligned} &\frac{1}{v}\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 \\ &\qquad = \int_0^\pi \Big([\mu_{\mathrm{eq}} + A(\xi^1)]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)] \Big)^{1/2} \sin(\xi^1)\cos(\xi^1) d\xi^1 \end{aligned} \tag{45}$$

with

$$\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 \;<\; 0 \tag{46}$$

since $\alpha, v > 0$. Equation (42) yields the maximum value $E_{\max}$ of E,

$$\begin{aligned} E^2_{\max} = 2mn_{\mathrm{ion}}\Bigg[-\mu_{\mathrm{eq}} + \mu_{\mathrm{wb}} + &\frac{v}{\int_0^\pi A(\xi^{1\prime})\sin(\xi^{1\prime})\cos(\xi^{1\prime}) d\xi^{1\prime}} \times \\ &\int_{\mu_{\mathrm{wb}}}^{\mu_{\mathrm{eq}}}\int_0^\pi \Big([\mu + A(\xi^1)]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)] \Big)^{1/2} \sin(\xi^1)\cos(\xi^1) d\xi^1\, d\mu \Bigg]. \end{aligned} \tag{47}$$

The above is a general expression for $E_{\max}$ given $A(\xi^1)$ as data. In the following, we determine a simple expression for an upper bound on $E_{\max}$ when $A(\xi^1) = -a\cos(\xi^1)$ where a is a positive constant ($a > 0$ ensures (46) is satified). Using (47) it follows

$$E^2_{\max} = 2mn_{\mathrm{ion}}\left\{ -\mu_{\mathrm{eq}} + \mu_{\mathrm{wb}} + \frac{3}{2}\frac{v}{a}\int_{\mu_{\mathrm{wb}}}^{\mu_{\mathrm{eq}}} [\mathcal{I}_+(\mu) + \mathcal{I}_-(\mu)]\, d\mu \right\}, \tag{48}$$

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[1] Note that the equilibrium of μ need not coincide with the plasma's thermodynamic equilibrium.

where

$$\mathcal{I}_{\pm}(\mu) = \pm \int_0^1 \Big([\mu \pm a\chi]^2 - \gamma^2[1 + R^2(1-\chi^2)] \Big)^{1/2} \chi\, d\chi \tag{49}$$

Furthermore, (45) may be written

$$\frac{3}{2}\frac{v}{a}[\mathcal{I}_+(\mu_{\mathrm{eq}}) + \mathcal{I}_-(\mu_{\mathrm{eq}})] = 1 \tag{50}$$

and since $\mathcal{I}_+(\mu_{\mathrm{eq}}) \geq \mathcal{I}_+(\mu)$ and $\mathcal{I}_-(\mu_{\mathrm{wb}}) \geq \mathcal{I}_-(\mu)$ for $\mu_{\mathrm{wb}} \leq \mu \leq \mu_{\mathrm{eq}}$, using (48, 50)

$$E^2_{\max} \leq \frac{3v}{a} m n_{\mathrm{ion}} (\mu_{\mathrm{eq}} - \mu_{\mathrm{wb}})[\mathcal{I}_-(\mu_{\mathrm{wb}}) - \mathcal{I}_-(\mu_{\mathrm{eq}})] \tag{51}$$

Furthermore $-\mathcal{I}_-(\mu_{\mathrm{eq}}) \leq \frac{1}{2}\sqrt{\mu^2_{\mathrm{eq}} - \gamma^2}$ and $\mathcal{I}_-(\mu_{\mathrm{wb}}) \leq 0$ so

$$E^2_{\max} \leq \frac{3v}{2a}\frac{m^2\omega_p^2}{q^2}(\mu_{\mathrm{eq}} - \mu_{\mathrm{wb}})\sqrt{\mu^2_{\mathrm{eq}} - \gamma^2} \tag{52}$$

where $\omega_p = \sqrt{n_{\mathrm{ion}} q^2/m}$ is the plasma angular frequency (in units where $\varepsilon_0 = 1$ and $c = 1$).

4·2. *Wave-breaking in an external magnetic field.* – In tackling (32), one may opt to seek only those V_ξ satisfying (33); this approach was followed in the preceeding sections. Although, at first sight, this method appears to be a simpler than attempting to solve (32), it is not always the simplest option. There are potential advantages in considering (32) in its generality, as we will now argue.

The component of the magnetic field parallel to the velocity of a point charge does not contribute to the Lorentz force on that point charge. A similar observation may also be applied to certain V_ξ in (32) even though the (∂_1, ∂_2) components of V_ξ are non-zero. Furthermore, the results of the previous section are unaffected by a constant magnetic field aligned along x^3.

The axially symmetric V_ξ introduced above is of the general form

$$V_\xi = (1 + Y^2 + Z^2)^{1/2}\partial_0 + Y\cos(\xi^2)\partial_1 + Y\sin(\xi^2)\partial_2 + Z\partial_3, \tag{53}$$

where $Y = \hat{Y}(x, \xi^1)$ and $Z = \hat{Z}(x, \xi^1)$. Suppose F is of the form

$$F = F_{\boldsymbol{I}} + F_{\boldsymbol{II}}, \tag{54}$$

$$F_{\boldsymbol{I}} = E(\zeta) dx^0 \wedge dx^3, \tag{55}$$

$$F_{\boldsymbol{II}} = B dx^1 \wedge dx^2 \tag{56}$$

We have

$$i_{V_\xi} F_{\boldsymbol{II}} = B\, Y \big(\cos(\xi^2) dx^2 - \sin(\xi^2) dx^1\big) \tag{57}$$

and furthermore

$$\frac{\partial \dot{\Sigma}^1}{\partial \xi^2} = -Y \sin(\xi^2), \tag{58}$$

$$\frac{\partial \dot{\Sigma}^2}{\partial \xi^2} = Y \cos(\xi^2) \tag{59}$$

so

$$g_{ab} \frac{\partial \dot{\Sigma}^a}{\partial \xi^2} dx^b = -Y \sin(\xi^2) dx^1 + Y \cos(\xi^2) dx^2. \tag{60}$$

Hence, using (26) it follows

$$\frac{q}{m} \iota_{V_\xi} F_{\boldsymbol{II}} \wedge \Omega_\xi = 0 \tag{61}$$

and from (32)

$$(\nabla_{V_\xi} \widetilde{V}_\xi - \frac{q}{m} \iota_{V_\xi} F_{\boldsymbol{I}}) \wedge \Omega_\xi = 0. \tag{62}$$

Therefore, if V_ξ satisfies (32) with $F = F_{\boldsymbol{I}}$, the same velocity field also satisfies (32) with $F = F_{\boldsymbol{I}} + F_{\boldsymbol{II}}$. It follows that a longitudinal magnetic field does not influence an axially symmetric discontinuity in the electron distribution and the results of the previous section hold for non-zero constant B.

5. – Beyond the Lorentz force

The previous discussion clearly shows that there is merit in considering (32) in its generality. We argue that for future extension of this work to fields with more complicated spacetime dependence, it is prudent to eschew (33) in favour of (32). We now illustrate this point further using a very simple example.

Let the chain Σ be such that

$$V_\xi = (1 + R^2)^{1/2} \partial_0 + R \sin(\xi^1) \cos(\xi^2) \partial_1 + R \sin(\xi^1) \sin(\xi^2) \partial_2 + R \cos(\xi^1) \partial_3 \tag{63}$$

where R is a function on $\mathcal{M}$ and, using (26),

$$\Omega_\xi = R^2 \sin(\xi^1) \cos(\xi^1) dx^1 \wedge dx^2 + R^2 \sin^2(\xi^1) dx^3 \wedge \Big(\sin(\xi^2) dx^1 - \cos(\xi^2) dx^2 \Big). \tag{64}$$

The 4-acceleration of V_ξ is

$$\begin{aligned} \nabla_{V_\xi} \widetilde{V}_\xi = &-\frac{R\, V_\xi R}{\sqrt{1 + R^2}}\, dx^0 \\ &+ V_\xi R \Big(\sin(\xi^1) \cos(\xi^2) dx^1 + \sin(\xi^1) \sin(\xi^2) dx^2 + \cos(\xi^1) dx^3 \Big) \end{aligned} \tag{65}$$

and, for simplicity, we assume that (65) is in response to a longitudinal electric field,

$$F = E dx^0 \wedge dx^3, \tag{66}$$

which contributes to (32) as

$$\iota_{V_\xi} F = E\Big((1+R^2)^{1/2} dx^3 - R\cos(\xi^1) dx^0\Big). \tag{67}$$

Clearly the Lorentz force equation (33) cannot be satisfied for general R, since $\nabla_{V_\xi}\widetilde{V}_\xi$ contains terms in dx^1, dx^2 which cannot cancel against terms in $\iota_{V_\xi} F$. However, the only nonzero contribution to the left-hand side of (32) can be made to vanish by requiring

$$V_\xi R = \frac{q}{m} E\sqrt{1+R^2}\cos(\xi^1). \tag{68}$$

Inspection of the ξ dependences in (63) and (68) reveals that R can depend only on x^3 and

$$\frac{d}{dx^3}\sqrt{1+R^2} = \frac{q}{m}E \tag{69}$$

so E also depends only on x^3.

To compare the above with solutions to (33), we seek a reparameterisation of Σ whose corresponding family of 4-velocities satisfies (33). In particular, we consider a map ρ

$$\begin{aligned} &\rho : \mathcal{V}\times\mathcal{D}' \to \mathcal{V}\times\mathcal{D} \\ &(x, \xi'^1, \xi'^2) \mapsto (x, \xi^1 = \psi^1(x,\xi'), \xi^1 = \psi^2(x,\xi')) \end{aligned} \tag{70}$$

where $\mathcal{V}\subset\mathcal{M}$ and $(\xi'^1,\xi'^2)\in\mathcal{D}'\subset\mathbb{R}^2$. Then given Σ satisfying $\Sigma^*\omega = 0$,

$$(\Sigma\circ\rho)^*\omega = \rho^*(\Sigma^*\omega) = 0. \tag{71}$$

The chains Σ and $(\Sigma\circ\rho)$ locally represent the same discontinuity, and are physically equivalent. However, the families $V_\xi = V_\xi^a\partial_a$ and $W_{\xi'} = W_{\xi'}^a\partial_a$ of vector fields, where

$$V_\xi^a = \Sigma^*\dot{x}^a, \qquad W_{\xi'}^a = (\Sigma\circ\rho)^*\dot{x}^a, \tag{72}$$

are different. We demand

$$\nabla_{W'_\xi}\widetilde{W}_{\xi'} = \frac{q}{m}\iota_{W_{\xi'}} F \tag{73}$$

and using (71) it follows

$$W_{\xi'}\psi^2 = 0, \qquad W_{\xi'}\psi^1 = -\frac{qE}{mR}\sqrt{1+R^2}\sin(\psi^1). \tag{74}$$

One may solve (74) to determine $(\Sigma\circ\rho)$, but is clear that (for general E) solving for the discontinuity in terms of Σ is a simpler task. Furthermore, we expect this state of affairs to hold for more complicated configurations.

6. – Conclusion

We have developed a covariant formalism for tackling discontinuities in 1-particle distributions. We have used it to develop wave-breaking limits for models of thermal plasmas whose distributions have effectively 1-dimensional spacetime dependence but are 3-dimensional in velocity.

Appendix A.

Vertical and horizontal lifts

Given tensors on a manifold $\mathcal{M}$, there are a number of ways to lift them onto the tangent manifold $T\mathcal{M}$. The simplest and best known of these are the vertical lift and (given a connection on $\mathcal{M}$) the horizontal lift. The following is a summary of some important properties of these lifts; for more details see, for example, [15].

The vertical lift is a tensor homomorphism. Acting on forms, it is equivalent to the pull-back with respect to the canonical projection map $\pi : T\mathcal{M} \to \mathcal{M}$:

$$\beta^{\boldsymbol{V}} = \pi^* \beta, \qquad \forall\ \beta \in \Lambda\mathcal{M}. \tag{A.1}$$

Acting on a vector Y, the vertical lift is

$$Y^{\boldsymbol{V}} = Y^a \frac{\partial}{\partial \dot{x}^a} \qquad \forall\ Y = Y^a \frac{\partial}{\partial x^a} \in T\mathcal{M}. \tag{A.2}$$

Note from these definitions, contraction of the vertical lift of a vector with the vertical lift of a form vanishes:

$$\iota_{(Y^{\boldsymbol{V}})} \beta^{\boldsymbol{V}} = 0. \tag{A.3}$$

The horizontal lift makes use of a connection ∇, with coefficients $\Gamma^a{}_{bc}$ in a coordinate basis $\partial_a = \partial / \partial x^a$:

$$\nabla_{\partial_c} \partial_b = \Gamma^a{}_{bc} \partial_a. \tag{A.4}$$

The horizontal lift $dx^{a\boldsymbol{H}}$ of the basis form dx^a is

$$dx^{a\boldsymbol{H}} = d\dot{x}^a + \dot{x}^c \Gamma^a{}_{bc}^{\boldsymbol{V}} dx^b, \tag{A.5}$$

while that of the basis vector $\partial / \partial x^a$ is

$$\left(\frac{\partial}{\partial x^a} \right)^{\boldsymbol{H}} = \frac{\partial}{\partial x^a} - \dot{x}^c \Gamma^b{}_{ac}^{\boldsymbol{V}} \frac{\partial}{\partial \dot{x}^b}. \tag{A.6}$$

The horizontal lifts of more general 1-forms and vectors may be determined from the relations

$$(f\beta)^{\boldsymbol{H}} = f^{\boldsymbol{V}} \beta^{\boldsymbol{H}} \qquad \beta \in \Lambda^1\mathcal{M}, \tag{A.7}$$

$$(fY)^{\boldsymbol{H}} = f^{\boldsymbol{V}} Y^{\boldsymbol{H}} \qquad \forall\ Y \in T\mathcal{M}, \tag{A.8}$$

for any function f on $\mathcal{M}$.

Similarly to the vertical lift, the contraction of the horizontal lift of a vector with the horizontal lift of a form vanishes:

$$\iota_{(Y^{\boldsymbol{H}})}\beta^{\boldsymbol{H}} = 0. \tag{A.9}$$

Two other useful identities relate to the contractions of vertical and horizontal lifts of forms and vectors:

$$\iota_{(Y^{\boldsymbol{V}})}\beta^{\boldsymbol{H}} = \iota_{(Y^{\boldsymbol{H}})}\beta^{\boldsymbol{V}} = (\iota_Y\beta)^{\boldsymbol{V}}. \tag{A.10}$$

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